# Blowing Bruce Willis Up...


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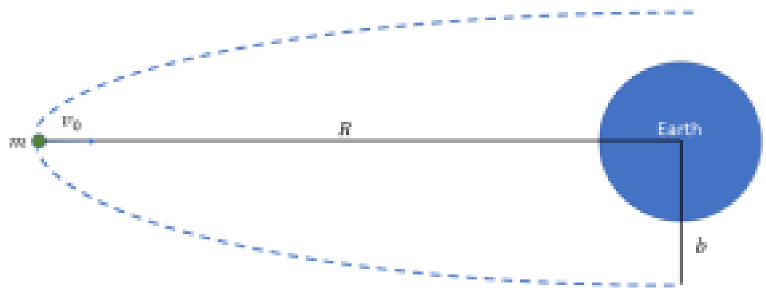


Figure 1: *R* is the distance of the asteroid from the earth's center, *b* is the impact parameter, *m* the original mass and $v_0$ the initial velocity,

For simplicity we assume that the initial velocity of the android is purely radial, i.e. left unchecked the asteroid will collide head-on.

We shall work backwards on this: assuming *b* is the minimal distance of the asteroid from the planet, at *r* = *b* we have:

$$\dot{r} = 0, \quad (1)$$

i.e. the radial velocity at that point is zero. This means that the asteroid travels exactly perpendicular to the line extending from the earth to *b*.

Now, we take energy calculations. The Hamiltonian (energy) of the asteroid is given by:

$$E(r, \theta, t) = \frac{m\dot{r}^2}{2} + \frac{mr^2\dot{\theta}^2}{2} - \frac{GmM}{r} \quad (2)$$

This is nothing but the kinetic energy decomposed into radial and tangential velocity components, and the gravitational energy of a point particle under some central gravitational potential.
Since $\theta$ is a cyclical coordinate (i.e. - no potential term involving $\theta$), the angular momentum is a conserved

quantity for this system. Thus we can write $L \equiv mr^2\dot\theta$, and substitute, to get:

$$E_{r,\theta,t} = \frac{m\dot r^2}{2} + \frac{L^2}{2mr^2} - \frac{GmM}{r}. \quad (3)$$

Due to the Hamiltonian, (well actually Lagrangian but who cares right now) not being explicitly time dependant we know the energy is conserved, thus we have:

$$E = E(r,\ \theta,\ \dot r,\ \dot\theta) \quad (4)$$

and

$$E\big|_{r=b} = E\big|_{r=R}. \quad (5)$$

All we have to do now is calculate a bit:

$$E\big|_{r=R} = \frac{mv_0^2}{2} + \frac{L^2}{2mR^2} - \frac{GmM}{R} = \quad (6)$$

$$= \frac{L^2}{2mb^2} - \frac{GmM}{b} = E\big|_{r=b}$$

A bit of algebra gets us:

$$\frac{L^2}{2m}\left(\frac{1}{b^2} - \frac{1}{R^2}\right) = \frac{mv_0^2}{2} + GmM\left(\frac{1}{b} - \frac{1}{R}\right) \quad (7)$$

$$\Rightarrow L^2\,\frac{R^2 - b^2}{2m\,b^2R^2} = \frac{mv_0^2}{2} + GmM\,\frac{R-b}{bR} \quad (8)$$

$$\Rightarrow L^2 = 2m\left(\frac{mv_0^2}{2}\,\frac{b^2R^2}{R^2-b^2} + GmM\,\frac{bR}{R+b}\right) \quad (9)$$

And we wish to bring it to a form we can easily Taylor expand:

$$\frac{L^2}{2mR^2} = \frac{mv_0^2}{2}\,\frac{b^2}{R^2}\left(\frac{1}{1+\frac{b}{R}}\right)\left(\frac{1}{1-\frac{b}{R}}\right) + \frac{GmM}{R}\,\frac{b}{R}\left(\frac{1}{1+\frac{b}{R}}\right). \quad (10)$$

So we got the necessary angular momentum one has to give half of an asteroid to clear the impact parameter $b$. Now let's look at the actual part of the explosion which is nothing but a plastic collision but backwards:

$$P_{\perp,init} = 0$$

$$P_{//,init} = mv_0 \quad (11)$$

$$\tilde m = \frac{m}{2} \quad (12)$$

Where $m$ is the original mass, and $\tilde m$ is the mass of each asteroid half post-explosion. As the ideal explosion gives the halves equal and opposite momenta we have post-explosion momenta:

$$P_\perp = \frac{m}{2} v_\perp \quad (13)$$

Since this is perpendicular additional velocity, all of the velocity goes to angular momentum, i.e.

$$L_1 = \frac{m v_\perp R}{2} \quad (14)$$

and the energy associated with this is given by:

$$\frac{L_1^2}{2\tilde{m} R^2}. \quad (15)$$

The only thing left to do is punch in the expression from Eq. (10) to get:

$$\Delta E = 2 \times \left[ \frac{\tilde{m} v_0^2}{2} \frac{b}{R} \left( \frac{1}{1+\frac{b}{R}} \right) \left( \frac{1}{1-\frac{b}{R}} \right) + \frac{G M \tilde{m}}{R} \frac{b}{R} \left( \frac{1}{1+\frac{b}{R}} \right) \right]. \quad (16)$$

Since $\tilde{m} = m/2$ we have:

$$\Delta E = \frac{m v_0^2}{2} \frac{b}{R} \left( \frac{1}{1+\frac{b}{R}} \right) \left( \frac{1}{1-\frac{b}{R}} \right) + \frac{G m M}{R} \frac{b}{R} \left( \frac{1}{1+\frac{b}{R}} \right). \quad (17)$$

Now, to Taylor expand the bejesus out of this, we assume $b \ll R$ and we will save expressions up to quadratic in $(b/R)$ :

$$\Delta E \simeq \frac{m v_0^2}{2} \frac{b}{R} \left(1-\frac{b}{R}\right)\left(1+\frac{b}{R}\right) + \frac{G m M}{R} \frac{b}{R} \left(1-\frac{b}{R}\right) \quad (18)$$

$$\simeq \frac{m v_0^2}{2} \frac{b}{R} + \frac{G m M}{R} \frac{b}{R} - \frac{G m M}{R} \left(\frac{b}{R}\right)^2 \quad (19)$$

It's nice to see, that the further you go, the less of a bang you have to have to give the same "buck"... Also you might want to drop the last term, as it is negligible with respect to the second term, though not necessarily with regards to the first term.
However, usually the kinetic and gravitational terms are of the same order of magnitude (Virial theorem? al though strictly speaking it doesn't apply here) so that's a good heuristic to justify dropping out the last term

2

anyhow...
Now all you have to do is substitute $R$ for the distance from earth the explosion occurs in, $b$ for the 'safe distance' you require and $m = \frac{4\pi D^3 \rho}{3}$, where $D$ is the asteroid's radius, $\rho$ is its (average) density. Also you should punch in Newton's universal gravitational constant $G$ and Earth's mass.

That would be the MINIMAL energy needed for an explosion that saves the earth....

Have fun in the next apocalypse....